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GENERALIZED POTENTIAL FLOW THEORY AND DIRECT CALCULATION OF VELOCITIES APPLIED TO THE NUMERICAL SOLUTION OF THE NAVIER–STOKES AND THE BOUSSINESQ EQUATIONS

WOUTER ZIJL

TNO-DGV Institute of Applied Geoscience, P.O. Box 285, 2600 AG Delft, The Netherlands

SUMMARY

A formulation based on three scalar functions or potentials is applied to analyse the Navier–Stokes and Boussinesq equations in three dimensions. In this formulation an explicit expression for the pressure exists, the so-called generalized Bernoulli equation. Therefore the scalar functions formulation may be considered as a generalization of the well-known potential flow and Bernoulli theory for irrotational fluid motion. The many advantages of this formulation applied to three-dimensional Navier–Stokes and Boussinesq flow will be discussed, and a numerical example is given as an illustration.

KEY WORDS Navier–Stokes Boussinesq Bernoulli Vorticity Potentials

1. INTRODUCTION

Since the advent of advanced computer architectures, namely, vector computers and parallel processors, numerical simulation of three-dimensional flow has become feasible. However, before developing numerical schemes to solve the three-dimensional equations, it will be advantageous to consider the possible formulations of the basic flow equations.

Three-dimensional numerical solutions to the Navier–Stokes equation governing the motion of fluids with constant density, and to the Boussinesq equation governing the motion of incompressible fluids in temperature and solute concentration fields, are conventionally based on either the formulation with the primitive variables velocity and pressure or on a formulation where the vorticity is made a primary variable. The disadvantage of the formulation based on the primitive variables arises from the difficulties in handling the incompressibility constraint. Disadvantages of a formulation based on the vorticity are that explicit boundary conditions for the vorticity are not available and that for three-dimensional calculations six unknowns are dealt with.

In this paper a formulation based on three scalar variables or potentials is presented. This formulation is basically equivalent to a formulation based on the vorticity but does not exhibit its disadvantages.

Since in this scalar representation an explicit expression for the pressure is available (the generalized Bernoulli equation), this formulation may be considered as a generalization of the well-known potential theory for irrotational flow. First the potential formulation for the Navier–Stokes equation is derived and then, on the basis of these former results, an extension to the Boussinesq equation is presented. Finally a procedure to compute the flow velocities very accurately is given.

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The conclusion is that a formulation based on scalar variables is very convenient for a theoretical and numerical analysis of flow phenomena. Also a numerical example dealing with Stokes flow will be presented.

2. MATHEMATICAL PRELIMINARIES

Theorem 1

This well-known theorem states that any irrotational vector field $\mathbf{u}$ may be represented by $\mathbf{u}=\nabla\phi$, where the scalar function ϕ is unique up to the addition of a constant.[1]

Theorem 2

Theorem 2, or Euler's theorem on divergence-free fields, states that any divergence-free vector field $\mathbf{w}$ may be represented by $\mathbf{w}=\nabla m\times\nabla\psi$, where m and ψ are scalar functions.[1]

Theorem 3

Since the curl of any vector field $\mathbf{v}$ is divergence-free, it follows from Theorem 2 that $\nabla\times\mathbf{v}=\nabla m\times\nabla\psi$. Furthermore, since $\nabla m\times\nabla\psi=\nabla\times(m\nabla\psi)$, it follows that $\nabla\times(\mathbf{v}-m\nabla\psi)=\mathbf{0}$. Consequently it follows from Theorem 1 that any vector field $\mathbf{v}$ may be written as $\mathbf{v}=\nabla\phi+m\nabla\psi$, where ϕ, m and ψ are scalar functions or potentials (sometimes called Monge potentials or Clebsch variables[1]).

3. THE BOUSSINESQ EQUATION IN POTENTIALS

The advective time derivative

In fluid kinematics the advective acceleration $\mathrm{D}\mathbf{v}/\mathrm{d}t$, i.e. the advective time derivative of the velocity field $\mathbf{v}$, plays an important part. The definition of $\mathrm{D}\mathbf{v}/\mathrm{D}t$ is given by

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{\partial\mathbf{v}}{\partial t}+(\nabla\times\mathbf{v})\times\mathbf{v}+\nabla(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}). \tag{1}$$

Use of Theorem 2 for $\mathbf{w}=\nabla\times\mathbf{v}$ results in

$$(\nabla\times\mathbf{v})\times\mathbf{v}=(\mathbf{v}\cdot\nabla m)\nabla\psi-(\mathbf{v}\cdot\nabla\psi)\nabla m.$$

Use of Theorem 3 for $\mathbf{v}$ results in

$$\frac{\partial\mathbf{v}}{\partial t}=\left(\frac{\partial m}{\partial t}\right)\nabla\psi-\left(\frac{\partial\psi}{\partial t}\right)\nabla m+\nabla\left(\frac{\partial\phi}{\partial t}+m\frac{\partial\psi}{\partial t}\right).$$

Substitution of these two expressions into the definition of the advective acceleration yields

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=\left(\frac{\mathrm{D}m}{\mathrm{D}t}\right)\nabla\psi-\left(\frac{\mathrm{D}\psi}{\mathrm{D}t}\right)\nabla m+\nabla\left(\frac{\partial\phi}{\partial t}+m\frac{\partial\psi}{\partial t}+\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right), \tag{2}$$

where

$$\frac{\mathrm{D}m}{\mathrm{D}t}=\frac{\partial m}{\partial t}+\mathbf{v}\cdot\nabla m,\qquad \frac{\mathrm{D}\psi}{\mathrm{D}t}=\frac{\partial\psi}{\partial t}+\mathbf{v}\cdot\nabla\psi$$

represent the advective time derivatives of the potentials m and ψ.

The viscous stress tensor

An important expression in fluid mechanics is the divergence $\nabla\cdot\mathbf{S}$ where $\mathbf{S}$ is given by

$$\mathbf{S}=\mu[\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}]+(k-\tfrac{1}{2}\mu)(\nabla\cdot\mathbf{v})\mathbf{I}. \tag{3}$$

Under the assumption that the dynamic viscosity μ and the bulk viscosity k are constant, the following expression is found:

$$\nabla\cdot\mathbf{S}=(k+\tfrac{4}{3}\mu)\nabla(\nabla\cdot\mathbf{v})-\mu\nabla\times(\nabla\times\mathbf{v}).$$

Use of Theorem 2 for $\mathbf{w}=\nabla\times\mathbf{v}$ yields

$$\nabla\times(\nabla\times\mathbf{v})=(\nabla^2\psi)\nabla m-(\nabla^2 m)\nabla\psi+(\nabla\psi\cdot\nabla)\nabla m-(\nabla m\cdot\nabla)\nabla\psi$$

and, as a result, the following expression is found:

$$\nabla\cdot\mathbf{S}=(\mu\nabla^2 m)\nabla\psi-(\mu\nabla^2\psi)\nabla m-\mu(\nabla\psi\cdot\nabla)\nabla m+\mu(\nabla m\cdot\nabla)\nabla\psi+\nabla[(k+\tfrac{4}{3}\mu)(\nabla\cdot\mathbf{v})]. \tag{4}$$

From the theory presented in the Appendix it follows that the vector field $(\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m$ is in the plane spanned by the vector fields ∇m and $\nabla\psi$, i.e.

$$(\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m=b\nabla\psi-\beta\nabla m, \tag{5a}$$

with

$$b=-\frac{\big([(\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m]\times\nabla m\big)\cdot(\nabla m\times\nabla\psi)}{(\nabla m\times\nabla\psi)\cdot(\nabla m\times\nabla\psi)}, \tag{5b}$$

$$\beta=-\frac{\big([(\nabla m\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\nabla m]\times\nabla\psi\big)\cdot(\nabla m\times\nabla\psi)}{(\nabla m\times\nabla\psi)\cdot(\nabla m\times\nabla\psi)}. \tag{5c}$$

Substitution of (5a) into (4) results in

$$\nabla\cdot\mathbf{S}=\mu(\nabla^2 m+b)\nabla\psi-\mu(\nabla^2\psi+\beta)\nabla m+\nabla[(k+\tfrac{4}{3}\mu)(\nabla\cdot\mathbf{v})]. \tag{6}$$

The Boussinesq equation

The equation of motion relating force and acceleration for a Newtonian fluid will be given by the Boussinesq equation

$$\mathrm{D}\mathbf{v}/\mathrm{D}t=\chi\mathbf{g}-\nabla\pi+\nabla\cdot(\mathbf{S}/\rho_0), \tag{7}$$

where $\mathbf{g}$ is the gravitational acceleration which is assumed to be irrotational, i.e. $\mathbf{g}=\nabla\Omega$; the fluid density ρ is given by $\rho=\rho_0(1+\chi)$, where ρ_0 is a constant density and χ describes the changes in density due to variations in temperature or concentration of dissolved mass;[2] $\pi=p/\rho_0-\Omega$, where p is the fluid pressure.

Substitution of expressions (2) and (6) into the equation of motion (7) results in

$$\left(\frac{\mathrm{D}m}{\mathrm{D}t}-\nu(\nabla^2 m+b)\right)\nabla\psi-\left(\frac{\mathrm{D}\psi}{\mathrm{D}t}-\nu(\nabla^2\psi+\beta)\right)\nabla m=\chi\mathbf{g}-\nabla\left(\frac{\partial\phi}{\partial t}+m\frac{\partial\psi}{\partial t}+\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{p}{\rho_0}-\Omega\right), \tag{8}$$

where $\nu=\mu/\rho_0$ is the kinematic viscosity. In agreement with the Boussinesq approximation it has been assumed in expression (8) that $\nabla\cdot\mathbf{v}=0$.

For reasons which will become clear in the next section, the curl of equation (8) will be taken, resulting in

$$\nabla\left(\frac{\mathrm{D}m}{\mathrm{D}t}-\nu(\nabla^2 m+b)\right)\times\nabla\psi-\nabla\left(\frac{\mathrm{D}\psi}{\mathrm{D}t}-\nu(\nabla^2\psi+\beta)\right)\times\nabla m=\nabla\chi\times\mathbf{g}. \tag{9}$$

4. FLUID FLOW WITH CONSTANT DENSITY

The Navier–Stokes equation

For flow where variations in density do not play a part, the Boussinesq equation (7) simplifies to the Navier–Stokes equation, where $\chi=0$. For $\chi=0$ it follows from equation (9) that

$$\mathrm{D}m/\mathrm{D}t-\nu(\nabla^2 m+b)-c_1\psi=c_2,$$

$$\mathrm{D}\psi/\mathrm{D}t-\nu(\nabla^2\psi+\beta)-c_3 m=c_4,$$

where $c_1(t)$, $c_2(t)$, $c_3(t)$ and $c_4(t)$ are integration constants that are a function of t only. The choice $c_1=0$, $c_3=0$, $c_4=0$, $c_2=\omega\neq 0$ yields the following two advection–diffusion equations:

$$\partial m/\partial t+\mathbf{v}\cdot\nabla m=\nu\nabla^2 m+\nu b+\omega, \tag{10a}$$

$$\partial\psi/\partial t+\mathbf{v}\cdot\nabla\psi=\nu\nabla^2\psi+\nu\beta. \tag{10b}$$

The reason for the choice $c_2=\omega\neq 0$ will be explained in the next section.

Substitution of equations (10a) and (10b) into equation (8) with $\chi=0$ results in an explicit expression for the pressure:

$$p=-\rho_0\left(\frac{\partial\phi}{\partial t}+m\frac{\partial\psi}{\partial t}+\omega\psi+\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}-\Omega\right)+c_5(t), \tag{11}$$

where the integration constant $c_5(t)$ is a function of t only. For constant $\mathbf{g}$, $\Omega=gz$, where z is the coordinate in the direction of $\mathbf{g}$.

For irrotational flow (i.e. $\nabla\times\mathbf{v}=\nabla m\times\nabla\psi=\mathbf{0}$) the term $m\partial\psi/\partial t+\omega\psi$ disappears and equation (11) simplifies to the well-known Bernoulli equation. For this reason equation (11) may be denoted as the generalized Bernoulli equation for rotational flow.

Boundary conditions

The boundary conditions are that

$$\tau_i\cdot\mathbf{v}=\frac{\partial\phi}{\partial\tau_i}+m\frac{\partial\psi}{\partial\tau_i},\qquad \mathbf{n}\cdot\mathbf{v}=\frac{\partial\phi}{\partial n}+m\frac{\partial\psi}{\partial n}$$

are specified on a closed boundary $\partial\mathrm{D}$, where $\mathbf{n}$ and τ_i ($i=1,2$) are the three unit vectors normal and parallel to the boundary respectively. The tangential boundary conditions on $\partial\mathrm{D}$ can be written as

$$m=\frac{\tau_1\cdot\mathbf{v}-\partial\phi/\partial\tau_1}{\partial\psi/\partial\tau_1}=\frac{\tau_2\cdot\mathbf{v}-\partial\phi/\partial\tau_2}{\partial\psi/\partial\tau_2}\quad\text{on }\partial\mathrm{D},$$

from which it follows that

$$\left(\tau_2\cdot\mathbf{v}-\frac{\partial\phi}{\partial\tau_2}\right)\frac{\partial\psi}{\partial\tau_1}-\left(\tau_1\cdot\mathbf{v}-\frac{\partial\phi}{\partial\tau_1}\right)\frac{\partial\psi}{\partial\tau_2}=0\quad\text{on }\partial\mathrm{D}.$$

On a non-moving no-slip boundary, where $\tau_i\cdot\mathbf{v}=0$, we find for ψ

$$\frac{\partial\phi}{\partial\tau_2}\frac{\partial\psi}{\partial\tau_1}-\frac{\partial\phi}{\partial\tau_1}\frac{\partial\psi}{\partial\tau_2}=0\quad\text{on }\partial\mathrm{D}.$$

If the ϕ-field is known, the boundary values of ψ can be determined by integration of this last expression. A possible choice is

$$\psi = \phi \quad \text{on } \partial D \tag{12a}$$

and the resulting boundary condition for m is

$$m = -1 \quad \text{on } \partial D. \tag{12b}$$

Boundary condition (12b) is possible thanks to the choice $c_2 = \omega \neq 0$ in equation (10a). If we had instead chosen $c_2 = 0$, we would find a non-trivial solution with $\partial m/\partial t \neq 0$, even for steady flow. Of course this would be very inconvenient for actual (numerical) calculations.

Substitution of boundary conditions (12a) and (12b) into the generalized Bernoulli equation (11) for the pressure on the boundary yields

$$p/\rho_0 - \Omega + \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + \omega\phi = 0 \quad \text{on } \partial D. \tag{12c}$$

From the above-presented theory it will be observed that the Navier–Stokes equation can be written as two coupled advection–diffusion equations (10a) and (10b) for which well-defined boundary conditions exist and where the pressure can simply be derived from the generalized Bernoulli equation (11). The two equations (10a) and (10b) will be coupled by a third equation for ϕ which will be discussed in Section 6.

The above description with potentials is very advantageous compared with the conventional description based on the vorticity. In this latter description a well-defined boundary condition for the vorticity does not exist, there is no simple way to determine the pressure and the three-dimensional formulation is based on six equations. In these respects it will be very attractive to base a (numerical) analysis of the Navier–Stokes equations on the above-presented formulation with potentials. However, an important question before making such a decision is whether the generalized potential flow theory can be extended to other types of flow than pure Navier–Stokes flow. For this purpose the extension of the generalized potential flow theory to flow governed by the Boussinesq equation will be discussed in the next section.

5. EXTENSION TO FLUIDS WITH VARYING DENSITY

Let us now assume that in equation (9) $\nabla \times \chi\mathbf{g} \neq \mathbf{0}$.

According to Theorem 3, the vector $\chi\mathbf{g}$ may be written as

$$\chi\mathbf{g} = \nabla\phi^* + m^*\nabla\psi^*. \tag{13}$$

Since ϕ^*, m^* and ψ^* are not uniquely defined by (13), we choose m^* such that ∇m^* is in the plane spanned by ∇m and $\nabla\psi$:

$$\nabla m^* = f_1\nabla m + f_2\nabla\psi. \tag{14}$$

We further choose a relationship between f_1 and f_2 given by

$$m\nabla f_1 + \psi\nabla f_2 = 0. \tag{15}$$

The choices (14) and (15) can be made without any loss of generality of expression (13), since ψ^* and f_1, $f_2(f_1)$ can be chosen to fit any arbitrary $\nabla \times \chi\mathbf{g} = f_1\nabla m \times \nabla\psi^* + f_2\nabla\psi \times \nabla\psi^*$ and ϕ^* can be chosen to fit any arbitrary $\nabla\cdot\chi\mathbf{g}$. Since both $\nabla \times \chi\mathbf{g}$ and $\nabla\cdot\chi\mathbf{g}$ completely determine the vector $\chi\mathbf{g}$ itself, the choices (14) and (15) do not limit the vector $\chi\mathbf{g}$ to a special choice.

The choices (14) and (15) result in $m^* = f_1 m + f_2\psi + \text{constant}$ and substitution of this result and expression (15) into (13) yields

$$\chi\mathbf{g} = \nabla(\phi^* + \text{constant} \times \psi^*) + m\nabla F_1 + \psi\nabla F_2, \tag{16a}$$

with

$$F_1 = f_1\psi^*, \qquad F_2 = f_2\psi^*. \tag{16b}$$

Taking the curl of expression (16a) and substituting the results into equation (9) yields

$$\nabla\left(\frac{Dm}{Dt} - \nu(\nabla^2 m + b) + F_2\right) \times \nabla\psi - \nabla\left(\frac{D\psi}{Dt} - \nu(\nabla^2\psi + \beta) - F_1\right) \times \nabla m = \mathbf{0}. \tag{17a}$$

Similar to equations (10a), (10b) we find

$$\partial m/\partial t + \mathbf{v}\cdot\nabla m = \nu\nabla^2 m + \nu b + \omega - F_2, \tag{18a}$$

$$\partial\psi/\partial t + \mathbf{v}\cdot\nabla\psi = \nu\nabla^2\psi + \nu\beta + F_1. \tag{18b}$$

Substitution of (18a) and (18b) into (8) and (16) results again in the generalized Bernoulli equation (11).

In this way we have obtained an extension of the Navier–Stokes theory presented in Section 4 at the expense of two additional unknowns F_1 and F_2 governed by equations which are coupled to the equations for the potentials m and ψ.

6. THE CONTINUITY EQUATION

The continuity equation consistent with the Boussinesq approximation is given by[2]

$$\nabla\cdot\mathbf{v} = 0. \tag{19}$$

Substitution of Theorem 3 into (19) yields

$$\nabla^2\phi = -\nabla\cdot(m\nabla\psi). \tag{20a}$$

On a non-moving no-slip boundary the boundary condition for Poisson equation (20a) for ϕ is given by either

$$\partial\phi/\partial n = \mathbf{n}\cdot\mathbf{v} + \partial\psi/\partial n \quad \text{on } \partial D \tag{20b}$$

(since $m = -1$ on ∂D; see equation (12b)), where $\mathbf{n}\cdot\mathbf{v}$ is the given inflow or outflow rate, or, according to equation (12c),

$$\phi = -\frac{1}{\omega}\left(\frac{p}{\rho_0} - \Omega + \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) \quad \text{on } \partial D, \tag{20c}$$

where p is the pressure specified on the boundary.

The coupled set of equations (15), (18a), (18b) and (20a) in domain D, together with the boundary conditions (12a), (12b) and either (20b) or (20c) on boundary ∂D, govern the flow in domain D. The velocity $\mathbf{v}$ follows from Theorem 3 by (numerical) differentiation: $\mathbf{v} = \nabla\phi + m\nabla\phi$.

7. DETERMINATION OF VELOCITY FIELD BY INTEGRATION

The vector Poisson equation for the velocity

Sometimes one is interested in a computational procedure where the velocity field $\mathbf{v}$ is calculated avoiding numerical differentiation. For instance, calculation without numerical differentiation of $\mathbf{v}$

is necessary when one is interested in the vertical velocity component under conditions where this component is very small with respect to the horizontal velocity component. Such a situation often occurs in shallow water or in the atmosphere. In this case we can find the solution $\mathbf{v}$ by numerical integration of the vector Poisson equation

$$\nabla^2\mathbf{v} = -\nabla\times(\nabla m\times\nabla\psi). \tag{21}$$

In the following discussion (21) will be proved and the correct boundary conditions for (21) will be derived.

With the vector expression $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$ we find from (21)

$$\nabla(\nabla\cdot\mathbf{v}) + \nabla\times(\nabla m\times\nabla\psi - \nabla\times\mathbf{v}) = \mathbf{0}. \tag{22}$$

Taking the divergence of (22) yields

$$\nabla^2(\nabla\cdot\mathbf{v}) = 0 \quad \text{in D}. \tag{23a}$$

In order to find $\nabla\cdot\mathbf{v} = 0$, we must specify one of the following boundary conditions to the Laplace equation (23a):[3]

(i) $\nabla\cdot\mathbf{v} = 0$ on ∂D; (23b)

(ii) $\partial(\nabla\cdot\mathbf{v})/\partial n = 0$ on ∂D (23c)

and, additionally, $\nabla\cdot\mathbf{v} = 0$ at at least one point on ∂D;

(iii) $\nabla\cdot\mathbf{v} = 0$ on ∂D_1, $\partial(\nabla\cdot\mathbf{v})/\partial n = 0$ on ∂D_2, (23d)
with $\partial D_1 \cup \partial D_2 = \partial D$.

Equation (23a) with either boundary condition (23b), (23c) or (23d) results in

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in D}. \tag{24}$$

Substitution of (24) into (22) yields

$$\nabla\times\mathbf{v} = \nabla m\times\nabla\psi + \nabla h, \tag{25}$$

where h is a scalar function. Taking the divergence of (25) yields a Laplace equation for h:

$$\nabla^2 h = 0. \tag{26a}$$

In order to find $\nabla\times\mathbf{v} = \nabla m\times\nabla\psi$ or $\nabla h = \mathbf{0}$, we must specify one of the following boundary conditions to the Laplace equation (26a):[3]

(i) $h = \text{constant}$ on ∂D; (26b)

(ii) $\partial h/\partial n = 0$ on ∂D; (26c)

(iii) $\partial h/\partial n = 0$ on ∂D_1, $h = \text{constant}$ on ∂D_2,
with $\partial D_1 \cup \partial D_2 = \partial D$. (26d)

Equation (26a) with either boundary condition (26b), (26c) or (26d) results in $\nabla h = \mathbf{0}$ in D or, equivalently,

$$\nabla\times\mathbf{v} = \nabla m\times\nabla\psi \quad \text{in D}. \tag{27}$$

From expressions (24) and (27) we observe that equation (21) with the above-presented boundary conditions give the correct velocity field $\mathbf{v} = \nabla\phi + m\nabla\psi$ with $\nabla\cdot\mathbf{v} = 0$.

The velocity boundary conditions

Normal flow boundary condition. Let us discuss further the meaning of the boundary conditions. From taking the dot product of expression (22) with the normal unit vector $\mathbf{n}$ it follows that boundary condition (23c) (or (23d) on ∂D_2) is equivalent to

$$\mathbf{n}\cdot\nabla\times(\nabla m\times\nabla\psi-\nabla\times\mathbf{v})=0 \quad \text{on } \partial D \text{ (or on } \partial D_2). \tag{28a}$$

Condition (28a) is satisfied when

$$\mathbf{n}\times(\nabla\times\mathbf{v})=\mathbf{n}\times(\nabla m\times\nabla\psi)$$

or, with

$$\mathbf{n}\times(\nabla m\times\nabla\psi)=\frac{\partial\psi}{\partial n}\nabla m-\frac{\partial m}{\partial n}\nabla\psi,$$

when

$$\mathbf{n}\times[\mathbf{n}\times(\nabla\times\mathbf{v})]=\frac{\partial\psi}{\partial n}\mathbf{n}\times\nabla m-\frac{\partial m}{\partial n}\mathbf{n}\times\nabla\psi \quad \text{on } \partial D \text{ (or on } \partial D_2).$$

On a non-moving no-slip boundary, where $\mathbf{n}\times\nabla m=\mathbf{0}$ and $\psi=\phi$, this results in the following expression for the tangential components of $\nabla\times\mathbf{v}$:

$$-\mathbf{n}\times[\mathbf{n}\times(\nabla\times\mathbf{v})]=\frac{\partial m}{\partial n}\mathbf{n}\times\nabla\phi \quad \text{on } \partial D \text{ (or on } \partial D_2). \tag{28b}$$

Condition (28b) states that the tangential components of $\nabla\times\mathbf{v}-\nabla m\times\nabla\psi$ are zero, which means the curl of this vector, $\nabla\times(\nabla\times\mathbf{v}-\nabla m\times\nabla\psi)$, has no normal component, i.e. (28a) is automatically satisfied by (28b).

Boundary condition (26b) (or (26d) on ∂D_2) is equivalent to $\mathbf{n}\times\nabla h=0$ on ∂D (or on ∂D_2) and from equation (25) it will be observed that this condition also results in boundary condition (28b).

In order to make the problem (21), (28b) well-posed, the boundary condition $\mathbf{n}\cdot\mathbf{v}$ specified on ∂D (or on ∂D_2) must be given too.[4]

Tangential flow boundary condition. It follows from (25) that boundary condition (26c) (or (26d) on ∂D_1) is equivalent to

$$\mathbf{n}\cdot\nabla\times\mathbf{v}=\mathbf{n}\cdot(\nabla m\times\nabla\psi)=\nabla\psi\cdot(\mathbf{n}\times\nabla m) \quad \text{on } \partial D \text{ (or on } \partial D_1). \tag{29a}$$

On a non-moving no-slip boundary, where $\mathbf{n}\times\nabla m=\mathbf{0}$, this results in $\mathbf{n}\cdot\nabla\times\mathbf{v}=0$. Again this condition is automatically satisfied when we specify

$$-\mathbf{n}\times(\mathbf{n}\times\mathbf{v})=\mathbf{0} \quad \text{on } \partial D \text{ (or on } \partial D_1). \tag{29b}$$

From (29b) it follows that boundary condition (26c) (or (26d) on ∂D_1) is just the no-slip condition on a non-moving boundary. Combination of boundary condition (29b) with boundary condition (23b) (or (23d) on ∂D_1) makes the partial differential system (21), (23b), (29b) equivalent to $\mathbf{v}=\nabla\phi+m\nabla\psi$, $\nabla\cdot\mathbf{v}=0$. The problem is well-posed if the part ∂D_1 of the boundary ∂D is connected.[4] If there exist more disconnected regions ∂D_1^1, ∂D_1^2, etc., the solution is not unique.

Summary

To determine the velocity field $\mathbf{v}$ by numerical integration instead of numerical differentiation, we have to solve the equation

$$\nabla^2\mathbf{v}=-\nabla\times(\nabla m\times\nabla\psi) \quad \text{in D.}$$

The boundary conditions for this equation are either

$\mathbf{n}\cdot\mathbf{v}$ is given on ∂D (or on ∂D_2),

$$-\mathbf{n}\times[\mathbf{n}\times(\nabla\times\mathbf{v})]=\mathbf{n}\times\frac{\partial m}{\partial n}\nabla\phi \quad \text{on } \partial D \text{ (or on } \partial D_2);$$

or

$$\mathbf{n}\times\mathbf{v}=\mathbf{0} \quad \text{on } \partial D \text{ (or on } \partial D_1),$$
$$\nabla\cdot\mathbf{v}=0 \quad \text{on } \partial D \text{ (or on } \partial D_1),$$

where ∂D is a non-moving no-slip boundary and ∂D_1 must be connected. Determination of the three velocity components by numerical integration has proved to be extremely accurate when dealing with flow in porous media (where the right-hand side of equation (21) is given by a different expression), even when the vertical component is orders of magnitude smaller than the horizontal velocity components.[5,6]

8. NUMERICAL EXAMPLE

As an example, two-dimensional steady Stokes flow of a fluid with constant density in a rectangular driven cavity will be considered, in which the upper boundary moves with velocity V in the horizontal x-direction.

For this problem the Boussinesq equation (7) simplifies to the Stokes equation

$$\nabla\cdot(\mathbf{S}/\rho_0)=\nabla\pi, \qquad \nabla\cdot\mathbf{v}=0. \tag{30}$$

Substitution of expression (4) into (30) yields

$$(v\nabla^2 m)\nabla\psi-(v\nabla^2\psi)\nabla m=v(\nabla\psi\cdot\nabla)\nabla m-v(\nabla m\cdot\nabla)\nabla\psi+\nabla\pi. \tag{31}$$

With $\nabla(\nabla m\cdot\nabla\psi)=(\nabla m\cdot\nabla)\nabla\psi+(\nabla\psi\cdot\nabla)\nabla m$ it follows from (31) that

$$(v\nabla^2 m)\nabla\psi-(v\nabla^2\psi)\nabla m=-2v(\nabla m\cdot\nabla)\nabla\psi+\nabla(\pi+v\nabla m\cdot\nabla\psi). \tag{32}$$

Let us now assume that $\psi=Vz$, which results in

$$(v\nabla^2 m)\nabla z=\nabla\left(\frac{\pi}{V}+v\frac{\partial m}{\partial z}\right). \tag{33}$$

Taking the curl of (33) yields

$$(v\nabla^2 m)\times\nabla z=\mathbf{0}. \tag{34}$$

Expression (34) can be satisfied if we choose

$$\nabla^2 m=0 \quad \text{in D}. \tag{35}$$

The generalized Bernoulli equation is

$$\frac{\pi}{vV}=-\frac{\partial m}{\partial z}. \tag{36}$$

From the continuity equation $\nabla\cdot\mathbf{v}$ and Theorem 3 we find

$$-\nabla^2\phi=V\frac{\partial m}{\partial z} \quad \text{in D}. \tag{37}$$

As boundary conditions for the Poisson equation (37) we choose

$$\phi = 0 \qquad \text{on } \partial D_1, \tag{38a}$$

$$\phi = Vx \qquad \text{on } \partial D_3, \tag{38b}$$

$$\partial\phi/\partial n = 0 \qquad \text{on } \partial D_2 \text{ and } \partial D_4. \tag{38c}$$

It follows then that the boundary conditions for equations (35) are given by

$$m = -\frac{1}{V}\frac{\partial \phi}{\partial z} \quad \text{on } \partial\Omega. \tag{39}$$

In all these expressions

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad \frac{\partial}{\partial n} = \pm \frac{\partial}{\partial x}.$$

As follows from Theorem 3 and the generalized Bernoulli equation (36), the physical interpretation of the results in dimensionless form is given by

$$(v_x^1, v_z^1) = \left(\frac{\partial \phi^1}{\partial x^1}, \frac{\partial \psi^1}{\partial z^1} + m\right), \tag{40a}$$

$$Re\left(p^1 - \frac{z^1}{Fr}\right) = -\frac{\partial m}{\partial z^1}. \tag{40b}$$

In equation (40a) and (40b) the dimensionless velocity $\mathbf{v}^1$ and pressure p^1 are made dimensionless with the driving velocity V and with ρV^2 respectively. The dimensionless space co-ordinates $\mathbf{r}^1$ are made dimensionless with the depth of the cavity L, while the Reynolds number Re and the Froude number Fr are defined as $Re = VL/\nu$ and $Fr = V^2/gL$. Finally the dimensionless potentials ϕ^1 and ψ^1 are made dimensionless with VL.

The set of dimensionless coupled equations was solved iteratively by successive substitutions, in which the solution of system (37), (38) is substituted into system (35), (39) and the solution of system (35), (39) is substituted into system (37), (38) and so on until a sufficient degree of convergence is reached. After 75 iterations a relative accuracy of 0·0001 was reached for ϕ and m. A rectangular mesh of nodal points with a mesh size of 1/32 was used in combination with conventional conforming linear triangular elements. The streamline pattern is shown in Figure 1 and the 'pressure' $Re(p^1 - z^1/Fr)$ is presented in Figure 2 in the form of isobars.

Comparison of Figure 1 with the numerical results obtained by Pan and Acrivos[7] shows that the agreement is good.

9. CONCLUSIONS

The analysis presented in this paper shows that theoretical analysis and numerical solution of the continuum equations governing the motion of fluids can conveniently be based on a formulation where the three Cartesian components v_x, v_y, v_z of the velocity field $\mathbf{v}$ have been replaced by three scalar fields or potentials ϕ, m, ψ. The advantages of a formulation with potentials are:

1. In the formulation based on the primitive variables the (from a practical point of view) most interesting physical quantities, velocity and pressure, are calculated directly by numerical integration and this is often considered as an advantage, especially in situations where free surfaces or fluid–structure interactions are important. The great disadvantage of this latter

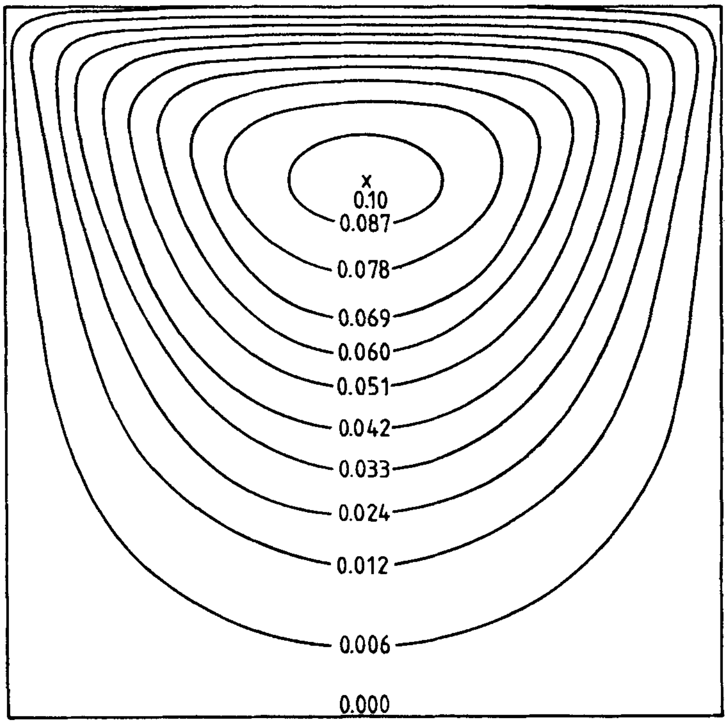

Figure 1. Streamline pattern; the numerical data represent the value of the stream function S, where $v_x^1 = -\partial S/\partial z^1$, $v_z^1 = \partial S/\partial x^1$

formulation arises from the difficulties in handling the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$. These difficulties do not arise in a formulation based on potentials and also then it is possible to compute one or more velocity components by numerical integration instead of numerical differentiation.

2. Transport of the two potentials m and ψ is accomplished by advection and viscous diffusion. Hence the range and timescale of significant variations in m and ψ is more restricted and slower than that of the primitive variables velocity and pressure, either of which can experience large changes being felt at infinity instantaneously. In a formulation with the potentials ϕ, m and ψ this latter behaviour is experienced by only one potential field, namely the generalized velocity potential ϕ satisfying a Poisson equation. As a result, in regions where the flow is irrotational this fact is automatically accounted for, resulting in a solution procedure based on classical, well-established potential flow theory for the classical velocity potential ϕ. Thus a formulation based on potentials is much 'milder' than a formulation based on the primitive variables velocity and pressure and consequently it may be expected to yield numerical results which are a better representation of the exact solution.
3. The two above-presented advantages are also valid for a conventional formulation based on the vorticity. However, if viscous flow is considered, explicit boundary conditions for the vorticity are not available, whereas explicit boundary conditions do exist for the potentials m and ψ representing the vorticity.
4. Another disadvantage of a conventional three-dimensional formulation based on the vorticity is that six unknowns (and consequently six equations) are present. A formulation in

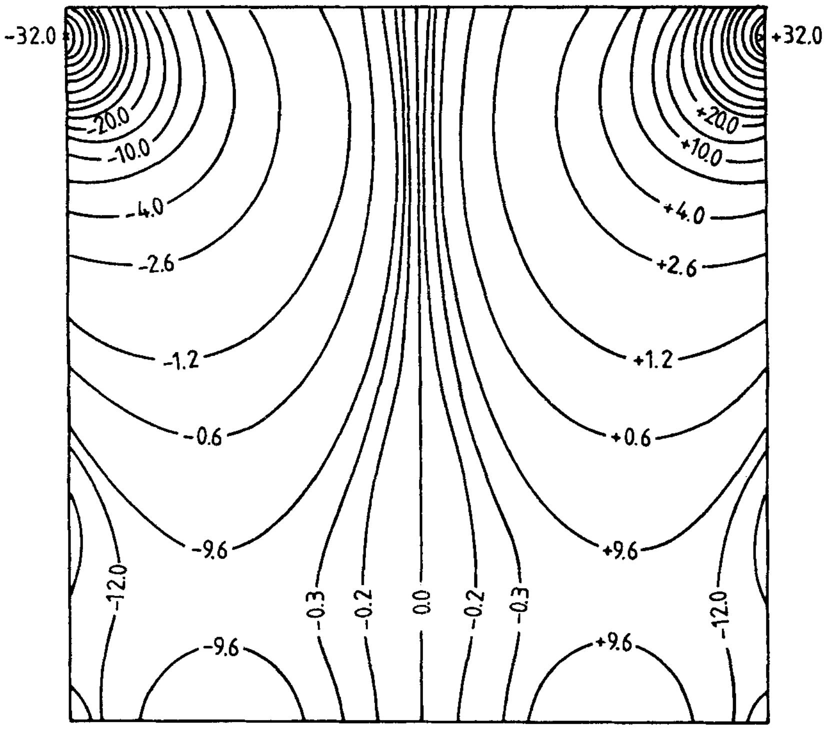

Figure 2. Isobar pattern; the numerical data represent the value of $Re(p^1 - z^1/Fr)$

potentials deals with only three variables (ϕ, m, ψ); thus it should lead to a more economical use of the computer in cases where three-dimensional fluid motion is considered.

5. In a conventional formulation based on the vorticity an explicit expression for the pressure does not exist. In the formulation with potentials an explicit expression for the pressure, the generalized Bernoulli equation, does exist. This is especially useful for the description of free surfaces and fluid–structure interactions.

NOMENCLATURE

b	function representing coupling between equations (m^{-2})
F_1	function representing density variations ($m^2\,s^{-2}$)
F_2	function representing density variations (s^{-1})
$\mathbf{g}$	gravitational acceleration ($m\,s^{-2}$)
k	fluid bulk viscosity (Pa s)
m	potential representing vorticity (dimensionless)
p	fluid pressure (Pa)
$\mathbf{v}$	fluid velocity ($m\,s^{-1}$)
x	horizontal Cartesian coordinate (m)
y	horizontal Cartesian coordinate (m)
z	vertical Cartesian coordinate (m)

Greek symbols

β	function representing coupling between equations (s^{-1})
μ	fluid dynamic viscosity (Pa s)
ν	fluid kinematic viscosity ($m^2 s^{-1}$)
π	$= p/\rho_0 - \Omega$ ($m^2 s^{-2}$)
ρ	fluid density ($kg\,m^{-3}$)
ρ_0	constant fluid density ($kg\,m^{-3}$)
ϕ	potential representing irrotational flow ($m^2 s^{-1}$)
χ	$=(\rho - \rho_0)/\rho_0$ (dimensionless)
ψ	potential representing vorticity ($m^2 s^{-1}$)
ω	function of time (s^{-1})
Ω	gravitational potential ($\mathbf{g} = \nabla\Omega$) ($m^2 s^{-2}$)

Other symbols

∇	gradient (m^{-1})
$\nabla\cdot$	divergence (m^{-1})
$\nabla\times$	curl (m^{-1})

APPENDIX

Consider the terms $\mathbf{a}\cdot\nabla\mathbf{b}$ and $\mathbf{b}\cdot\nabla\mathbf{a}$. In general orthogonal curvilinear co-ordinates these terms are given by

$$(\mathbf{a}\cdot\nabla\mathbf{b})_j = \frac{a_i}{h_i}\frac{\partial b_j}{\partial \zeta_i} + \frac{b_i}{h_i h_j}\left(a_j \frac{\partial h_j}{\partial \zeta_i} - a_i \frac{\partial h_i}{\partial \zeta_j}\right) \quad (j \text{ not summed}), \tag{41}$$

$$(\mathbf{b}\cdot\nabla\mathbf{a})_j = \frac{b_i}{h_i}\frac{\partial a_j}{\partial \zeta_i} + \frac{a_i}{h_i h_j}\left(b_j \frac{\partial h_j}{\partial \zeta_i} - b_i \frac{\partial h_i}{\partial \zeta_j}\right). \tag{42}$$

Subtraction of equation (42) from equation (41) yields

$$(\mathbf{a}\cdot\nabla\mathbf{b} - \mathbf{b}\nabla\mathbf{a})_j = \frac{1}{h_i}\left(a_i \frac{\partial b_j}{\partial \zeta_i} - b_i \frac{\partial a_j}{\partial \zeta_i}\right) + \frac{1}{h_i h_j}\frac{\partial h_j}{\partial \zeta_i}(b_i a_j - a_i b_j). \tag{43}$$

Now choose the unit vector $\mathbf{e}_1$ in the direction of the vector $\mathbf{a}$, i.e.

$$\mathbf{a} = a_1 \mathbf{e}_1. \tag{44}$$

The vector $\mathbf{b}$ consists of a component in the direction of $\mathbf{e}_1$ and a component normal to $\mathbf{e}_1$, i.e. in the direction of the unit vector $\mathbf{e}_2$:

$$\mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2. \tag{45}$$

Since $a_3 = 0$, $b_3 = 0$ it follows from expression (43) that

$$(\mathbf{a}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{a})_3 = 0. \tag{46}$$

Consequently $\mathbf{a}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{a}$ is in the plane spanned by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, which also means that $\mathbf{a}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{a}$ is in the plane spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.

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